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Image analysis of two point sources

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Abstract. In this paper the images of two point sources are considered. The effects of diffraction are discussed, and techniques for reducing the effects of a very bright point source on the image of a much fainter point source are suggested. The implications of these ideas on the prospects of obtaining images of extra solar planets are also discussed.

The image of a point source formed by an optical instrument is affected by the diffraction that arises at the entrance aperture of this instrument. In practice one observes the diffraction pattern of the point. If two light sources are present the radiation from one of the sources may prevent the second source from being observed, especially when one of the light sources is much brighter than the other one. In this paper some methods based on Fourier analysis for extracting the information on a point source in the vicinity of a much brighter source are discussed. This is then applied to the study of the motion of radiation of an extra solar planet which is in close proximity to its parent star by considering the motion of a localized skewing in the intensity distribution of the diffraction pattern of the parent star.

The diffraction pattern of a uniformly illuminated circular aperture that is impinged upon by a distant monochromatic point light source is known as an Airy pattern. The analytic form of the intensity distribution is given by

$$I = C^2 A^2 \left[\frac{2J_1(ka\eta)}{ka\eta} \right]^2 \tag{1}$$

where C is a constant and A is the area of the aperture. The constant C can be expressed in terms of the energy incident on the aperture per unit time, E:

$$C = \frac{1}{\lambda}\sqrt{\frac{E}{A}} \tag{2}$$

where λ is the wavelength of the monochromatic radiation that illuminates the circular aperture, and the wave number k is given by

$$k = \frac{2\pi}{\lambda}\,. \tag{3}$$

a is the radius of the aperture, η is defined by the expression

$$\eta = \sin(\theta) \tag{4}$$

(Born and Wolf 1970).

Consider the function $g(x' - x_0, y' - y_0)$. Its Fourier transform can be expressed as

$$G(f_x, f_y) = \exp[\mathrm{j}([f_x x_0 + f_y y_0)] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x', y') \exp(-\mathrm{j}[f_x x' + f_y y']) \, \mathrm{d}x' \, \mathrm{d}y' . \tag{5}$$

If $g(x' - x_0, y' - y_0)$ is a radially symmetric function its Fourier transform can also be expressed as

$$G(f_r, \theta) = 2\pi \exp[\mathrm{j}\{f_x x_0 + f_y y_0\}] \int_0^{\infty} J_0(r' f_r) \hat{g}_R(r') r' \, \mathrm{d}r' \tag{6}$$

where

$$\hat{g}_R(r') = g(x' - x_0, y' - y_0) . \tag{7}$$

$\hat{g}_R(r')$ is a radially symmetric function:

$$G(f_r, \theta) = \{2\pi \cos(f_r r_0 \cos[\phi - \theta_0]) + \mathrm{j}2\pi \sin(f_r r_0 \cos[\phi - \theta_0])\} \int_0^{\infty} J_0(r' f_r) g_R(r') r' \, \mathrm{d}r' . \tag{8}$$

According to equation (8) the imaginary component of the Fourier transform of the classical diffraction pattern of a circular aperture is zero when the coordinates of the origin are at the position of the classical intensity maximum of a point source. If instead of one point there are two distant monochromatic point sources present, the intensity distribution of the image of the second point will not be radially symmetrical when the plane polar coordinates, (r', θ') have the position of the largest classical intensity maximum of the brighter point source as the origin. The Fourier transform of the diffraction pattern of the second point will not be symmetric in f_r and it will also have a non-zero imaginary component.

The above discussion demonstrates that, in principle, the diffraction patterns of two classical light sources can be separated by taking the Fourier transform of the two diffraction patterns and analysing the real component of the Fourier transform at a constant value of f_r and separating the component that varies with the angle ϕ from the component that has a fixed value. From this expression one can then obtain the function $\hat{g}_R(r')$ corresponding to the two point sources by utilizing the inverse Fourier transform.

Alternatively one can obtain $\hat{g}_R(r')$ by utilizing the imaginary component of the Fourier transform of the intensity.

An example of a system where there is a large difference in intensity between two sources is a star and an extra solar planet that orbit their mutual centre of gravity. The image contrast between the planet and the star that it is associated with is about 10^9 at optical wavelengths and 10^7 at infrared wavelengths (Burke *et al* 1993, Watson *et al* 1991). From the above discussion the classical problem is readily solved. However, in practice one has to take quantum effects due to fluctuations in photon intensity into consideration. The variation in the photon number will, of course, result in the intensity of the primary star having a large component that is not radially symmetric. The intensity of the photon noise at the position of the largest intensity maximum of the planet Airy pattern may be much greater than the intensity of the radiation due to the planet. The magnitude of the photon noise at a point is dependent on the average (mean) rate at which photons arrive at that point. The fluctuations in the photon number at a point can, to a good degree of approximation, be modelled by the Poisson distribution (Dereniak and Crowe 1984) where the probability of obtaining m photons in a time t can be expressed as

$$p(m, t) = \frac{(\lambda t)^m}{m!} \exp(-\lambda t) \tag{9}$$

where λ is the average number of photons that arrive in the region of interest per unit time, and t is the time. The mean of the Poisson distribution is equal to λt and the variance is also equal to λt (Buckingham 1983).

Techniques have been developed to reduce the intensity of the diffracted starlight at the position of the planet. This reduction in the stellar photon number will have the beneficial effect of reducing the magnitude of the fluctuations in stellar intensity due to the photon noise. The diffracted light due to the brighter star can be suppressed by apodizing the energy in its diffraction pattern or by using modern interferometric technology (Shao 1991, Burke *et al* 1993). Burke *et al* (1993) gave details of a study using the apodization approach where they obtained a reduction in the intensity of light from the central star by a factor of 10^{-3} by using a Lyot filter, and high quality optics. Research reported by Shao (1991) demonstrated that reductions of the intensity of starlight at the position of a planet of the order of 10^{-3} were possible, in principle, with the interferometric approach.

The fraction of the stellar luminosity incident on a square region of side 2Δ centred on the point (u_0, v_0) is given by

$$L = \frac{1}{\pi} \int_{u_0-\Delta}^{u_0+\Delta} \mathrm{d}u \int_{v_0-\Delta}^{v_0+\Delta} \frac{J_1^2(\sqrt{u^2+v^2})}{u^2+v^2}\, \mathrm{d}v \qquad (10)$$

where u and v are the plane Cartesian coordinates of the centre of a square region in the image plane and

$$u = ka\eta_x \qquad (11a)$$

$$v = ka\eta_y \qquad (11b)$$

$$\eta = \sqrt{(\eta_x^2 + \eta_y^2)} \qquad (11c)$$

$$\Delta = ka\eta\,. \qquad (11d)$$

Consider a square region which has the maximum intensity of the diffraction pattern of an extra solar planet at its centre. The number of photons that originate from the primary star will, in general, be much larger than the number of photons incident on this square region from the planet. The photon number from both of these sources in a given time interval fluctuates. A comparison of the fluctuation of the photon number associated with a square region of side 2Δ when the only source of photons incident on this region is the primary star with a similar region which also has a contribution from an extra solar planet shows that the presence of the planet can have a significant skewing effect on the photon number; see figures 1 and 2. The bright region in the vicinity of the primary diffraction maximum of the planet will also move according to Keplerian dynamics. This suggests that one can obtain evidence about dim secondary objects that are associated gravitationally with the primary star if one can locate a bright region that is close to the primary star that moves according to Keplerian dynamics.

A method has been proposed (Lawunmi 1995a) that can analyse the radial velocity of a bright primary star to check for the presence of low-mass secondary objects that are associated gravitationally with the bright primary star. If the bright region that is near to the primary star moves according to Keplerian dynamics this method can be used to analyse the motion of the candidate secondary object and can obtain orbital parameters for this object from data that spans a fraction of its orbital period. The analysis of the Keplerian motion of a secondary source should allow the identification of the dim secondary object that accompanies a bright primary source within one quarter of the orbital period. The movement of the secondary source due to the mutual gravitational attraction of the components of the binary system is much greater than the corresponding motion of the

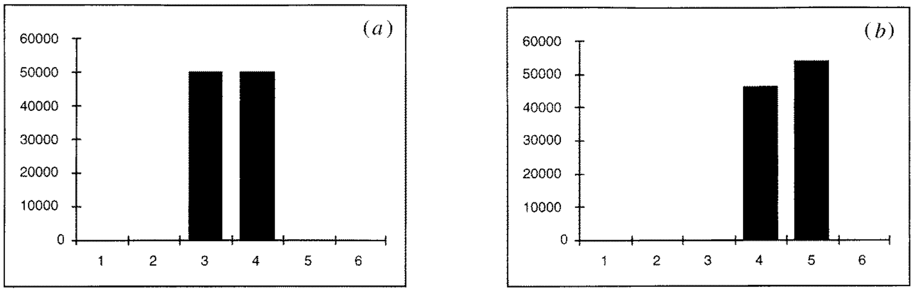

Figure 1. The following histograms illustrate the skewing effects that the presence of an extra solar planet can have on the contribution of the photon noise to the diffraction pattern of the primary star. This is done by considering the impact of the photons from an extra solar planet on a small square region of the diffraction pattern of a bright primary star. They measure the fluctuation in the photon number about the mean number of photons obtained in a classical diffraction pattern. The photons were counted over a square region of side 2Δ ($2\Delta = 7.50$ units) over a period of ten hours. In histogram (*a*) the only source of radiation that is incident on the square region is a bright star. In histogram (*b*) radiation from a composite source consisting of a bright star and a much fainter secondary source is incident on the square region. In both of these histograms 10^5 calculations of the deviation of the photon number from the mean number were used. The total system efficiency in this analysis was assumed to be 0.2. The primary star is 5 pc away from the observer and emits 10^9 photons s m^2 in the vicinity of the observer's telescope. The analysis presented in these figures is based on a monochromatic light source with a wavelength of 6000 Å. This should be applicable to the analysis of polychromatic light sources though the spread in the wavelengths may require one to measure the effects of light from a prospective planet from a larger area.

The square region is centred on a point that is 1 arcsec away from the position of the central diffraction maximum of the primary star. The planet is a Jupiter-like planet. The observer obtains 4 photons/s/m^2 due to the planet (Burke *et al* 1993). The telescope that was used for these calculations had a diameter of 2 m. In (*b*) the introduction of photons from the Jupiter-like planet has a marked skewing effect on the fluctuation in the photon number that is obtained in this region.

Flutuations due to Poisson noise were simulated 10^5 times in order to model the effects of light from the primary star on the light from the planet. N represents the difference between the number of photons detected in the square region and the mean stellar photon number obtained for classical diffraction from light due to the primary star in this region. Column 1 represents the number of times that $N < -1.5 \times 10^5$; column 2 represents the number of times that $-1.5 \times 10^5 \leqslant N < -7.5 \times 10^4$; column 3 represents the number of times that $-7.5 \times 10^4 \leqslant N < 0$; column 4 represents the number of times that $0 \leqslant N < 7.5 \times 10^4$; column 5 represents the number of times that $7.5 \times 10^5 \leqslant N < 1.5 \times 10^5$; column 6 represents the number of times that $N > 1.5 \times 10^5$.

primary source. In practice this means that the orbital parameters that are derived from analysing the secondary source are less prone to noise than the corresponding parameters that are obtained from the primary source.

Consider a binary system. The distance of the secondary object from the barycentre of the binary system, r', can be expressed as

$$r' = \left(\frac{m^*}{m_S + M^*}\right) r \tag{12}$$

where r is the distance of the centre of gravity of the primary object from the centre of gravity of the secondary object. The semimajor axis of the orbit of the secondary object

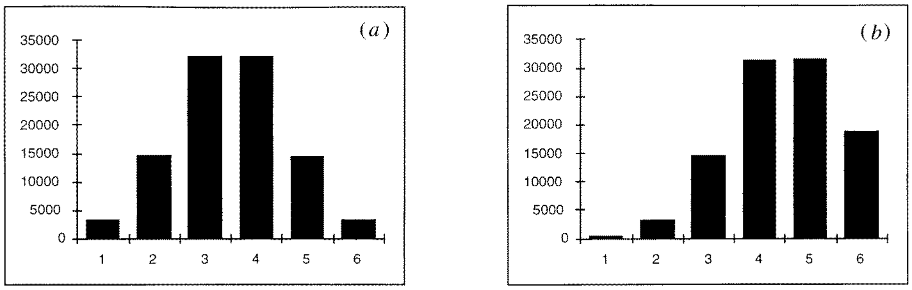

Figure 2. (See the first paragraph of the caption to figure 1.) The square region is centred on a point that is 0.2 arcsec away from the position of the central diffraction maximum of the primary star. The planet is an 'Earth-like' planet. The observer obtains 1 photon/s/m^2 due to the planet (Burke *et al* 1993). The telescope that was used for these calculations had a diameter of 4 m. A larger telescope was used in these calculations to increase the number of photons that could be obtained from the planet within a ten hour time interval. The light from the star on the square region was reduced by a factor of 10^{-3} in order to curtail the magnitude of the fluctuation in the number of stellar photons in the vicinity of the central part of the diffraction pattern of the planet. The reduction in the number of stellar photons in the locality of the planet allows the central core planetary diffraction pattern to have a significant skewing effect on the photon count within the square region of interest. N represents the difference between the number of photons detected in the square region and the mean stellar photon number obtained for classical diffraction from light due to the primary star in this region. Column 1 represents the number of times that $N < -1.5 \times 10^5$; column 2 represents the number of times that $-1.5 \times 10^5 \leqslant N < -7.5 \times 10^4$; column 3 represents the number of times that $-7.5 \times 10^4 \leqslant N < 0$; column 4 represents the number of times that $0 \leqslant N < 7.5 \times 10^4$; column 5 represents the number of times that $7.5 \times 10^5 \leqslant N < 1.5 \times 10^5$; column 6 represents the number of times that $N > 1.5 \times 10^5$.

with respect to its barycentre a' can be written as

$$a' = \left(\frac{m^*}{m_S + M^*}\right) a \tag{13}$$

where a is the semimajor major axis of the relative orbit of the centre of gravity of the secondary object with respect to the primary object. The motion of the secondary object across the celestial sphere can be described by utilizing the Thiele–Innes method (Green 1985). Its motion on the celestial sphere consists of two perpendicular components which can be described by the following equations:

$$x = X_1(\cos(E) - e) + X_2\sqrt{1 - e^2}\sin(E) + n_x(E) \tag{14}$$

and

$$y = Y_1(\cos(E) - e) + Y_2\sqrt{1 - e^2}\sin(E) + n_y(E) \tag{15}$$

where $n_x(E)$ and $n_y(E)$ are functions that represent the effect of the noise on the astrometric motion of the secondary object.

The quantities X_1, X_2, Y_1 and Y_2, are related to the orbital parameters of the binary system:

$$X_1 = a'(\cos(\varpi)\sin(\Omega) + \sin(\varpi)\cos(\Omega)\cos(i)) \tag{16}$$

$$X_2 = a'(-\sin(\varpi)\sin(\Omega) + \cos(\varpi)\cos(\Omega)\cos(i)) \tag{17}$$

$$Y_1 = a'(\cos(\varpi)\cos(\Omega) - \sin(\varpi)\sin(\Omega)\cos(i)) \tag{18}$$

$$Y_2 = a'(-\sin(\varpi)\cos(\Omega) - \cos(\varpi)\sin(\Omega)\cos(i)) \tag{19}$$

where i is the orbital inclination, ϖ is the longitude of periastron and Ω is the position angle of ascending nodes.

The Thiele–Innes constants can be used to obtain a', i, $(\varpi + \Omega)$, and $(\varpi - \Omega)$ unambiguously from the following equations (20)–(25); however, further information is required to determine ϖ and Ω uniquely.

$$X_1 - Y_2 = a' \sin(\varpi + \Omega)(1 + \cos(i)) \tag{20}$$

$$X_1 + Y_2 = -a' \sin(\varpi - \Omega)(1 - \cos(i)) \tag{21}$$

$$X_2 - Y_1 = -a' \cos(\varpi - \Omega)(1 - \cos(i)) \tag{22}$$

$$X_2 + Y_1 = a' \cos(\varpi + \Omega)(1 + \cos(i)) \tag{23}$$

and by noting that both

$$1 + \cos(i) \geqslant 0 \tag{24}$$

and

$$1 - \cos(i) \geqslant 0\,. \tag{25}$$

The radial velocity method of Lawunmi (1995) can be adapted to determine sets of orbital parameters corresponding to (T, e, E_0), where E_0 is the value of the eccentric anomaly corresponding to the initial data point. If it is possible to obtain a consistent set of orbital parameters for the astrometric data, the Thiele–Innes constants can be extracted with this technique. The Thiele–Innes constants can then be utilized to determine the mass of the secondary object and the inclination of the orbits of the primary and secondary objects with respect to their barycentre. The proposed method for analysing the astrometric motion of a bright region in the vicinity of the primary star is able to pick out Keplerian motion when the secondary object has not completed an orbital cycle.

In order to obtain high-quality images from Earth-bound telescopes it is necessary to reduce the deleterious effects of turbulence in the atmosphere on the resolution of the telescope. Many large telescopes are being fitted with adaptive optical systems. These systems measure the wavefront phase errors that arise from atmospheric turbulence. A deformable component of the adaptive system adjusts its shape in response to the effects of turbulence on the point source. This allows the telescope to compensate for the rapid fluctuations and movements of cells of air in the atmosphere.

The adaptive optical system produces intensity fluctuations. These intensity fluctuations show speckle structure that arises from interference patterns that develop during a cycle of adaptive correction. These intensity patterns accumulate over many cycles of adaptive correction to produce a random effect which hinders the detection of an extra solar planet (Angel 1994). When analysing images from telescopes fitted with an adaptive optical system for extra solar planets the effects of speckle noise have to be considered in addition to those that arise from photon noise. Angel (1994) demonstrated with the adaptive optical system that he proposed that the effects of speckle noise can be reduced to a degree that is sufficient to allow the telescope to obtain an image of an extra solar planet. The restrictions on the quality of the optical components of a telescope that is being used to scan the vicinity of the brighter primary source to check for a small bright region that is undergoing Keplerian motion are in general not as strict as the corresponding restrictions that are applicable when a single high quality image of a very faint point source is required. Angel's work suggests that it is feasible for an Earth-bound telescope that utilizes an appropriate adaptive optical system to gather data for the hybrid Keplerian-analysis–direct-imaging detection method that is being proposed in this paper.

At infrared wavelengths the ratio of the planet flux to the stellar flux is expected to be much higher than the corresponding ratio that is obtained from the visible region of the electromagnetic spectrum. This ratio is expected to be of the order of $1{:}10^7$ (Watson *et al* 1991). This improved image contrast is counterbalanced to some extent as there are problems associated with detecting faint secondary sources in the infrared region of the spectrum that are not encountered in the visible region of the spectrum such as thermal noise and zodiacal light from the star–planet system. The optics of a ground-based telescope can emit enormous quantities of infrared radiation. Thermal radiation that originates from the telescope optics can swamp a signal from a weak source. This difficulty can be reduced by maintaining the temperature of the infrared telescope at a low value ($\leqslant 70$ K) (Burke *et al* 1993).

The effects of zodiacal light due to dust in the planet–stellar system are dependent on the location and distribution of the dust relative to the planet. If the intensity of the infrared radiation due to the zodiacal light does not undergo large and rapid changes (of the order of the magnitude of the intensity of the infrared radiation due to the planet) along the path of the orbit of the planet, it should be possible to detect a planet using the method proposed in this paper.

In addition to these effects it is also important to consider the impact of defects in the telescope optics and alignment errors on the efficacy of the proposed method (Shao 1991, Brown and Burrows 1990). Surface roughness causes light to be scattered from the central region of the stellar diffraction pattern into the outer regions of the stellar diffraction pattern. This will increase the number of stellar photons relative to planetary photons. It will also introduce intensity fluctuations due to photon noise which will result in a further reduction in the number of planetary photons relative to stellar photons. The effects of surface roughness can undermine an attempt to extract a direct image of an extra solar planet from a region that is close to the primary star. The optical criteria for the hybrid direct-imaging–astrometric-analysis method are not as stringent as those for direct imaging as the objective is to measure a statistical increase in brightness that moves according to Keplerian dynamics. This method does not necessarily require the optics of the telescope to be of a sufficiently high quality to form a single sharp image of the planet. A more detailed analysis of the impact of scattering and aberrations on imaging extra solar planets is given by Brown and Burrows (1990) and Lawunmi (1996).

Conclusions

A hybrid direct-imaging–astrometric-analysis method of extra solar planet detection has been proposed in this article. It was shown that an extra solar planet can have a significant impact on a small area of the diffraction pattern of the primary star. This enhanced area of brightness should move according to Keplerian dynamics. If the region associated with the extra solar planet is sufficiently bright the Keplerian motion associated with the bright region should help to distinguish it from a fluctuation in brightness that is caused by stellar photon noise. The optical quality of the components of the telescope that are required to obtain a bright region due to the planet may not need to be as high as those that are required to obtain a single sharp direct image of the planet. This is significant as super-smooth optics are required to reduce the effects of scattering to a level that will allow a telescope to obtain a direct image of an extra solar planet. Manufacturing high-quality optical components that will allow a direct image to be obtained is currently problematic techologically (Shao 1991, Burke *et al* 1993).

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